THERMAL CONDUCTIVITY OF LAMINATED COMPOSITE FINITE BODIES OF COMPLEX CROSS SECTION TAKING THERMAL CONTACT RESISTANCES INTO ACCOUNT

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A new analytical method of solving the three-dimensional stationary problem of the thermal conduction of laminar composite finite bodies of complex cross section when there is imperfect thermal contact between the layers is proposed, which is based on the simultaneous use of integral transformations for each layer, a structural method, and the Bubnov—Galerkin method.

The thermal conduction of complex composite finite bodies (when there is imperfect thermal contact between the layers) is of considerable practical importance, since heat transfer in thermally protected constructions when calculating electrical and thermal insulation, thermal fields in elements of electronic equipment, and many other cases require the solution of such a problem.

There are practically no analytical methods of solving three-dimensional problems of thermal conduction for laminar composite bodies of complex cross section because of a number of mathematical difficulties of a fundamental nature, which arise due to the need to take into account the geometry of the region considered (an element of the construction), and the interaction between an element and the surrounding medium (the matching conditions at the boundary).

We will consider composite elements of spatial constructions in the form of multilayer bodies (the contact surface $z = a_k$) of complex transverse cross section, bounded by the planes $z = 0$ and $z = d$, and a cylindrical surface of complex form S, the generatrices of which are perpendicular to the planes $z = \text{const}$ (the regions $\Omega = \bigcup_{k=0}^{m} \Omega_k$).

Suppose that the determination of the temperature field in a given layer of the composite body reduces to solving the three-dimensional heat-conduction problem

$$\operatorname{div}(\lambda_k \operatorname{grad} T_k) = -W_k, \tag{1}$$

$$\left(\frac{\partial T_m}{\partial z} + h_m T_m\right)\Big|_{z=a_m} = 0, \quad \left(-\frac{\partial T_0}{\partial z} + h_0 T_0\right)\Big|_{z=0} = 0, \tag{2}$$

$$\lambda_k \frac{\partial T_k}{\partial z}\Big|_{z=a_{k+1}} = \frac{1}{R_k}(T_{k+1} - T_k)\Big|_{z=a_{k+1}}, \quad \lambda_k \frac{\partial T_k}{\partial z}\Big|_{z=a_{k+1}} = \lambda_{k+1}\frac{\partial T_{k+1}}{\partial z}\Big|_{z=a_{k+1}}, \tag{3}$$

$$T_k\Big|_{S_{1k}} = \varphi_{1k}, \quad \lambda_k \frac{\partial T_k}{\partial \nu_{2k}}\Big|_{S_{2k}} = q_{2k}, \tag{4}$$

where φ_{1k} and q_{2k} are continuous functions specified on the surface S_{1k} and S_{2k}.

We will introduce the integral transformation

$$u(x, y, \gamma) = \int_0^{a_m} T(x, y, z) N(\gamma, z) \lambda(z)\, dz = \sum_{k=0}^{m-1} \int_{a_k}^{a_{k+1}} T_k(x, y, z) \lambda_k(z) N_k(\gamma, z)\, dz, \tag{5}$$

where the kernels of the integral transformation $N_k(\gamma, z)$ are found by solving the equation

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$$\frac{d^2N_k(\gamma, z)}{dz^2} + \frac{1}{\lambda_k(z)} \frac{d\lambda_k(z)}{dz} \frac{dN_k(\gamma, z)}{dz} + \gamma^2 N_k(\gamma, z) = 0 \tag{6}$$

for uniform boundary conditions (2) and (3) and for $\lambda_k = \text{const}$ have the form

$$N_k(\gamma, z) = C_{1k} \sin \gamma x + C_{2k} \cos \gamma x. \tag{7}$$

Here $C_{11} = 1$, the coefficients C_{1k} and C_{2k} are determined taking into account the uniform conditions (2) and (3), and the eigennumbers γ_n are found from the corresponding characteristic equation.

Applying the integral transformation (5) to Eq. (1) and using the boundary conditions (4), we obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \gamma^2 u = -\overline{W}; \tag{8}$$

$$u\Big|_{S_1} = \overline{\varphi}_1; \quad \frac{\partial u}{\partial \nu_2}\Big|_{S_2} = \overline{q}_2, \tag{9}$$

where

$$\overline{\varphi}_1 = \sum_{k=0}^{m-1} \int_{a_k}^{a_{k+1}} \lambda_k \varphi_{1k} N_k(\gamma, z)\, dz; \quad \overline{q}_2 = \sum_{k=0}^{m-1} \int_{a_k}^{a_{k+1}} q_{2k} N_k(\gamma, z)\, dz;$$

$$\overline{W} = \sum_{k=0}^{m-1} \int_{a_k}^{a_{k+1}} W_k N_k(\gamma, z)\, dz.$$

The structure of the solution of problem (8) and (9), which accurately satisfies conditions (9), can be written in the form [1]

$$u(x, y, \gamma) = \Phi_0(x, y, \gamma) + \sum_{i,j} C_{ij}(\gamma)\, X_{ij}(x, y), \tag{10}$$

where, according to [2, 3],

$$\Phi_0 = (\overline{\varphi}_1 \omega_2^2 + \overline{q}_2 \omega_1)(\omega_1 + \omega_2^2)^{-1};$$

$$X_{ij} = \omega_1 \varphi_{ij} - \omega \left[\frac{\partial(\omega_1 \varphi_{ij})}{\partial x} \frac{\partial \omega_2}{\partial x} + \frac{\partial(\omega_1 \varphi_{ij})}{\partial y} \frac{\partial \omega_2}{\partial y}\right];$$

$$\omega = \omega_1 + \omega_2 - \sqrt{\omega_1^2 + \omega_2^2}.$$

The functions ω_1 and ω_2 are constructed using R-functions [2] and satisfy the conditions

$$\omega_1|_{S_1} = 0; \quad \omega_1 > 0 \ (x, y) \in \Omega;$$

$$\omega_2\Big|_{S_2} = 0; \quad \omega_2 > 0 \ (x, y) \in \Omega; \quad \frac{\partial \omega_2}{\partial \nu_2}\Big|_{S_2} = 1.$$

The Bubnov—Galerkin system for finding the undetermined coefficients $C_{ij}(\gamma)$ of the structure of solution (10) has the form [1]

$$\sum_{i+j=0}^{n} (A_{ijks} + \gamma^2 B_{ijks})\, C_{ij}(\gamma) = E_{ks}(\gamma), \tag{11}$$

where $k + s = 0, 1, \ldots, n$;

$$A_{ijks} = \iint_{\Omega_1} \Delta X_{ij} X_{ks}\, d\Omega_1; \quad B_{ijks} = -\iint_{\Omega_1} X_{ij} X_{ks}\, d\Omega_1;$$

$$E_{ks}(\gamma) = \iint_{\Omega_1} [-\overline{W} - \Delta\overline{\Phi}_0 + \gamma^2 \overline{\Phi}_0]\, X_{ks}\, d\Omega_1.$$

We will obtain the solution of problem (1)-(4) in the form

$$T(x, y, z) = \sum_{n=1}^{\infty} u(x, y, \gamma_n)\, C_{\gamma_n} N(\gamma_n, z), \tag{12}$$

where

$$C_{\gamma_n} = \left[\sum_{k=0}^{m-1} \int_{a_k}^{a_{k+1}} \lambda_k N_k^2(\gamma_n, z)\, dz\right]^{-1}.$$

For example, for a two-layer body ($m = 2$, and $z = a_1$ is the contact surface of the layers) with

$$T_1|_{z=0} = T_2|_{z=a_2} = 0 \tag{13}$$

the kernels of the integral transformation (5) have the form

$$N_1(\gamma, z) = \sin \gamma z\, [R\lambda_1\gamma \cos \gamma a_1 + \sin \gamma a_1] = \sin \gamma z \beta_1(\gamma),$$
$$N_2(\gamma, z) = \sin \gamma (a_2 - z) \sin \gamma (a_2 - a_1).$$

The characteristic equation for determining the eigennumbers γ_n has the form

$$\lambda_1 \gamma \cos \gamma a_1 \sin \gamma (a_2 - a_1) + \lambda_2 \gamma \cos \gamma (a_2 - a_1)\, \beta_1(\gamma) = 0.$$

For this case, with $R_k = 0\,(k = 1, 2)$ and $m = 3$ (a three-layer body of complex cross section) the kernels of the integral transformation (5) have the form

$$N_1(\gamma, z) = (\cos \gamma a_1 + f_3 \sin \gamma a_1) \sin \gamma z\, (\sin \gamma a_1)^{-1},$$
$$N_2(\gamma, z) = \cos \gamma z + f_3 \sin \gamma z,$$
$$N_3(\gamma, z) = (\cos \gamma a_2 + f_3 \sin \gamma a_2) \sin \gamma (a_3 - z) [\sin \gamma (a_3 - a_2)]^{-1},$$
$$f_3 = [\lambda_2 \gamma f_1 \cos \gamma a_1 - \sin \gamma a_1][\cos \gamma a_1 + \lambda_2 \gamma f_1 \sin \gamma a_1]^{-1},$$
$$f_1 = \sin \gamma a_1 (\lambda_1 \gamma \cos \gamma a_1)^{-1}.$$

The eigennumbers γ_n can be found by solving the characteristic equation

$$(\cos \gamma a_1 + \lambda_2 \gamma f_1 \sin \gamma a_1)(\lambda_2 \gamma f_2 \cos \gamma a_2 - \sin \gamma a_2) = (\lambda_2 \gamma f_1 \cos \gamma a_1 - \sin \gamma a_1)(\cos \gamma a_2 + \lambda_2 \gamma f_2 \sin \gamma a_2),$$

where

$$f_2 = \sin \gamma (a_2 - a_3) [\lambda_3 \gamma \cos \gamma (a_3 - a_2)]^{-1}.$$

For problem (1)-(4) (a two-layer body of complex cross section, $R_1 = 0$) the kernels of the integral transformation (5) have the form

$$N_1(\gamma, z) = \frac{(\cos \gamma a_1 + \beta \sin \gamma a_1)(\cos \gamma z + h_1 \gamma^{-1} \sin \gamma z)}{(\cos \gamma a_1 + h_1 \gamma^{-1} \sin \gamma a_1)},$$
$$N_2(\gamma, z) = \cos \gamma z + \beta \sin \gamma z,$$

where

$$\beta = (\gamma \sin \gamma a_2 - h_2 \cos \gamma a_2)(h_2 \sin \gamma a_2 + \gamma \cos \gamma a_2)^{-1},$$

and the eigennumbers γ_n are the roots of the equation

$$\frac{\lambda_1}{\lambda_2}(-\gamma \sin \gamma a_1 + h_1 \cos \gamma a_1)(\cos \gamma a_1 + \beta \sin \gamma a_1) = (-\gamma \sin \gamma a_1 + \beta \gamma \cos \gamma a_1)(\cos \gamma a_1 + h_1 \gamma^{-1} \sin \gamma a_1).$$

For a two-layer body of complex cross section with $\lambda_1 = \text{const}$, $\lambda_2 = \lambda_{20} z$, $R_1 = 0$, $T_1|_{z=0} = T_2|_{z=a_2} = 0$ the kernels of the integral transformation (5) have the form

$$N_1(\gamma, z) = \sin \gamma z,$$
$$N_2(\gamma, z) = \sin \gamma a_1 [J_0(\gamma, z) + \beta Y_0(\gamma, z)][J_0(\gamma a_1) + \beta Y_0(\gamma a_1)]^{-1},$$

where

$$\beta = -J_0(\gamma a_2)[Y_0(\gamma a_2)]^{-1},$$

and the eigenvalues γ_n are found by solving the characteristic equation

$$\frac{\lambda_{20}}{\lambda_1} \sin \gamma a_1 [J_0'(\gamma a_1) + \beta Y_0'(\gamma a_1)] = \gamma \cos \gamma a_1 [J_0(\gamma a_1) + \beta Y_0(\gamma a_1)].$$

Integral transformation (5) enables the heat-conduction problem for a laminated composite body of complex transverse cross section to be reduced to a much simpler problem in image space, the solution of which is found by the combined use of the structural method and the Bubnov—Galerkin method.

The proposed method enables one to obtain a solution of the three-dimensional heat-conduction problem for laminated composite bodies of complex transverse cross section with a high degree of accuracy, since in this case it is sufficient to obtain solutions of the corresponding two-dimensional heat-conduction problems in image space with a specified accuracy. This makes this method an effective one for analyzing heat-conducting systems with irregular geometry.

NOTATION

$h_m = \alpha_m \lambda_m^{-1}$; $h_0 = \alpha_0 \lambda_0^{-1}$; α_m, α_0, heat-transfer coefficients; λ_k, thermal conductivity of the k-th layer; R_k, thermal resistances of the k-th contact; W_k, a function characterizing the power of the internal sources of heat of the k-th layer; and Ω_1, transverse cross section of the body.

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